3.6.1 Special Matrix Forms

An interesting type of matrix that we did not discuss before is the **idempotent matrix**. This is a matrix that has the multiplication property

$$\mathbf{X}\mathbf{X} = \mathbf{X}^2 = \mathbf{X}$$

and therefore the property

$$\mathbf{X}^n = \mathbf{X}\mathbf{X}\cdots\mathbf{X} = \mathbf{X}, \quad n \in \mathcal{I}^+$$

(i.e., n is some positive integer). Obviously the identity matrix and the zero matrix are idempotent, but the somewhat weird truth is that there are lots of other idempotent matrices as well. This emphasizes how different matrix algebra can be from scalar algebra. For instance, the following matrix is idempotent, but you probably could not guess so by staring at it:

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & 2 \\ 4 & -4 & 4 \end{bmatrix}$$

(try multiplying it). Interestingly, if a matrix is idempotent, then the difference between this matrix and the identity matrix is also idempotent because

$$(\mathbf{I} - \mathbf{X})^2 = \mathbf{I}^2 - 2\mathbf{X} + \mathbf{X}^2 = \mathbf{I} - 2\mathbf{X} + \mathbf{X} = (\mathbf{I} - \mathbf{X}).$$

We can test this with the example matrix above:

$$(\mathbf{I} - \mathbf{X})^2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & 2 \\ 4 & -4 & 4 \end{bmatrix} \right)^2$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}^2 = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}.$$

Relatedly, a square **nilpotent** matrix is one with the property that $\mathbf{X}^n = \mathbf{0}$, for a positive integer n. Clearly the zero matrix is nilpotent, but others exist as